Quantum-sound Property Tests for Linear and Affine Linear Functions

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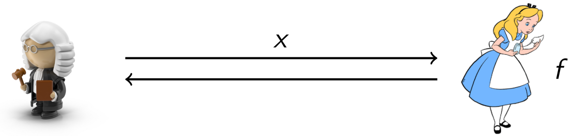

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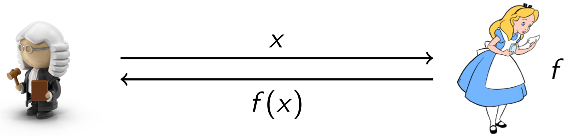

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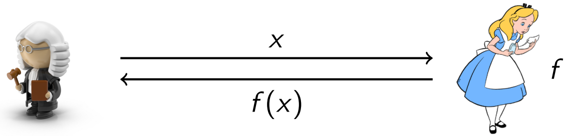

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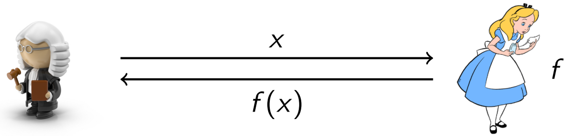
- We can use randomness to verify if the prover is actually saying what they have.



The BLR Test

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Let $f : \mathbb{F}_2^n \to \mathbb{F}_2$ be a function.

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- It is much faster, but it only tests approximate linearity.

$$\text{Ex. } \mathrm{odd}_5'(x) = \mathrm{odd}_5(x) \ \ \forall x \neq 00000, \ \mathrm{odd}_5'(00000) = 1$$

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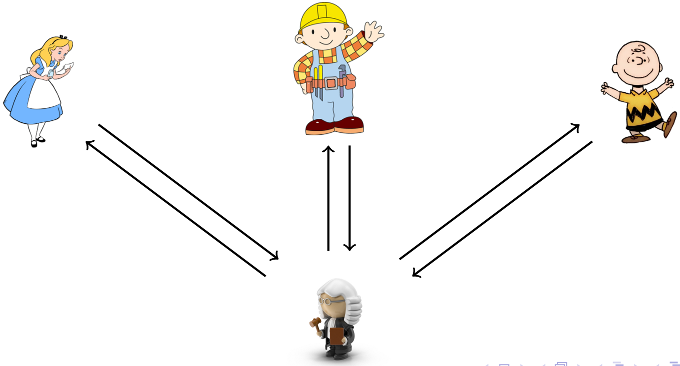

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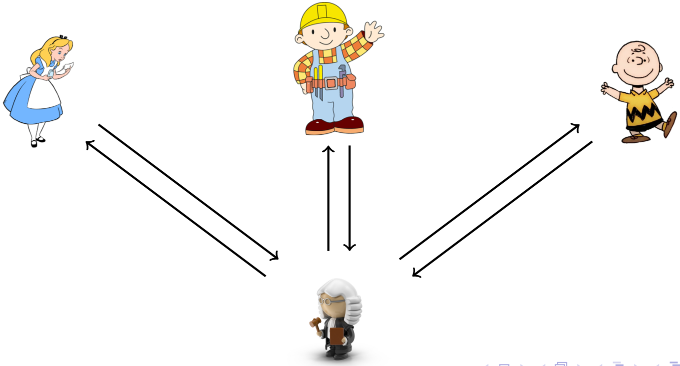

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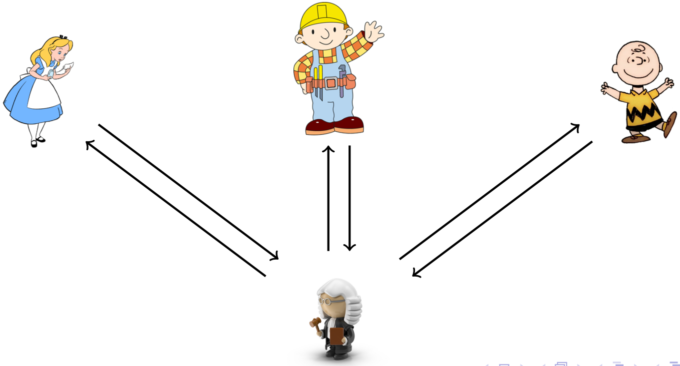

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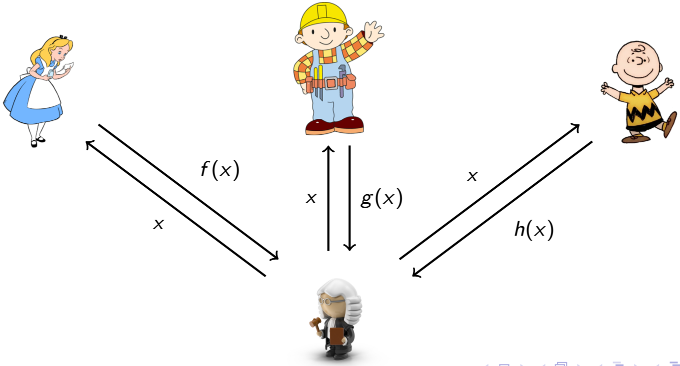

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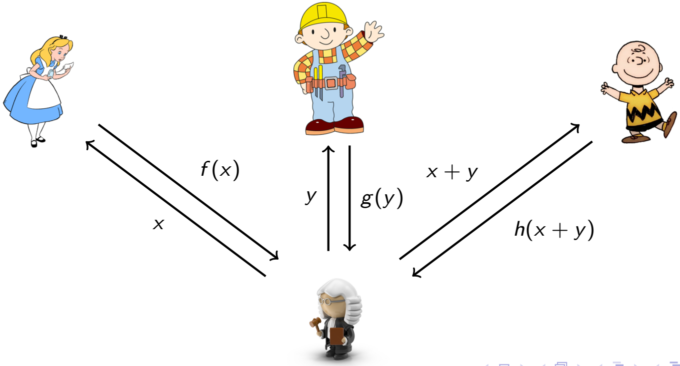

The Distributed BLR

More formally:

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The provers can have a number of different strategies that they may use.

Strategies

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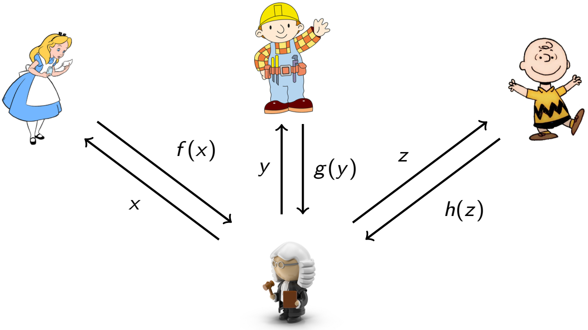

Strategies (cont.)

- A **strategy with shared randomness** is a probabilistic mixture of deterministic strategies given by $\{(p(\lambda), f_\lambda, g_\lambda, h_\lambda)\}_\lambda$ for $f_\lambda, g_\lambda, h_\lambda : \mathbb{F}_2^n \to \mathbb{F}_2$ and $p(\lambda) \in [0, 1]$.

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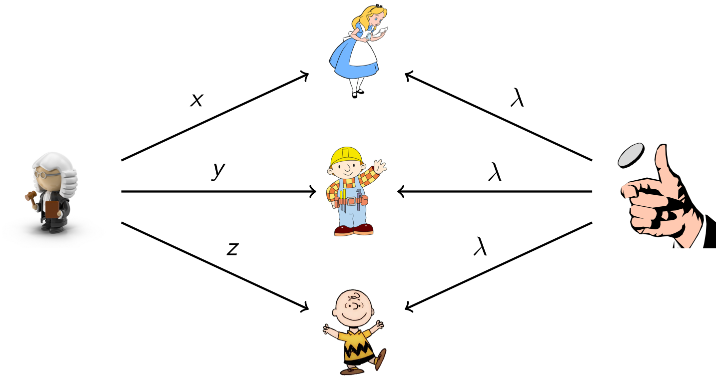

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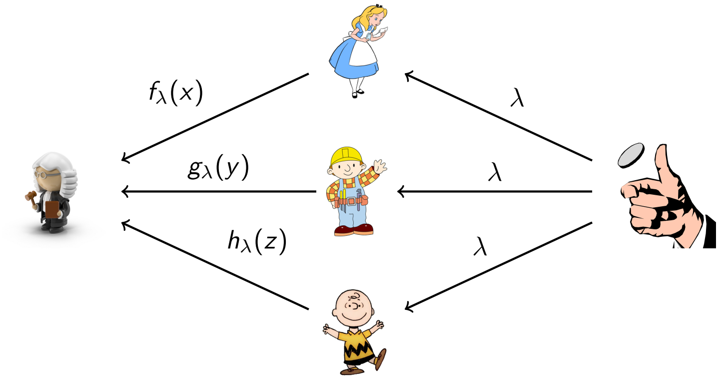

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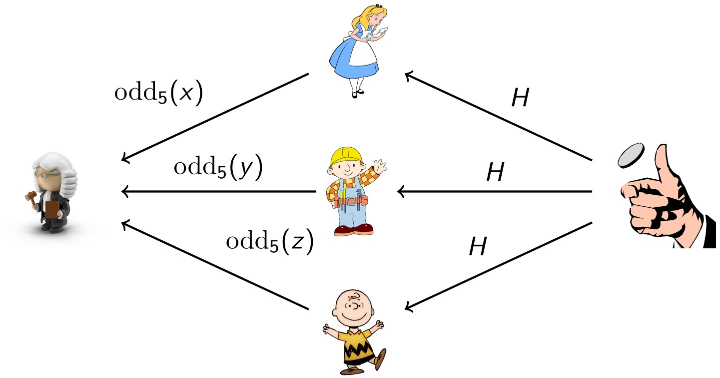

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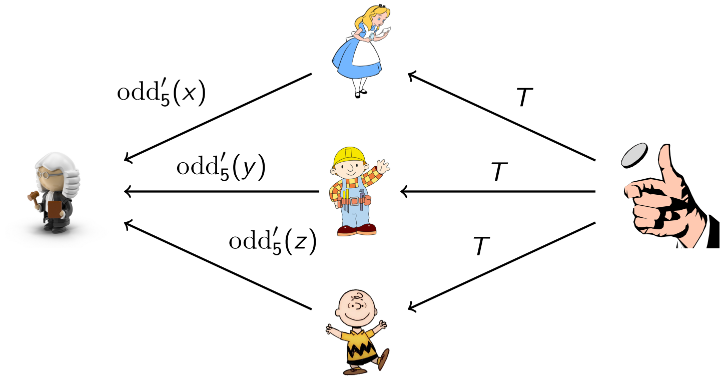

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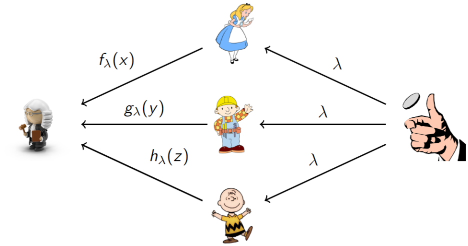


- We may model this as a joint probability distribution over the outputs:

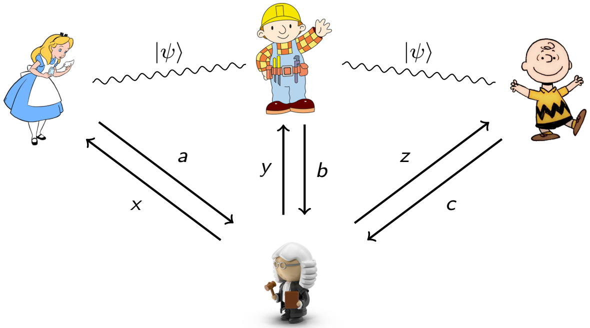
$$p(a, b, c|x, y, z) = \sum_\lambda p(\lambda)\delta_{f_\lambda(x)=a}\delta_{g_\lambda(y)=b}\delta_{h_\lambda(z)=c}$$

where $\delta_{j=k} = 1$ if $j = k$ and 0 otherwise.

Strategies (cont.)

- An **entangled strategy** $(A, B, C, |\psi\rangle)$ is a strategy where the players each have measurements $\{A_x^a\}$, $\{B_y^b\}$ and $\{C_z^c\}$ and share entanglement, or a state $|\psi\rangle$. They make measurements on the state depending on x, y and z, deciding the distribution:

$$p(a, b, c|x, y, z) = \langle\psi|A_x^a \otimes B_y^b \otimes C_z^c|\psi\rangle$$



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The **winning probability** of a strategy is

$$\frac{1}{2} \mathop{\mathbb{E}}_{x,y \sim \mathbb{F}_2^n} \left[\sum_{a,b \in \mathbb{F}_2} p(a, b, a+b | x, y, x+y) \right] + \frac{1}{2} \mathop{\mathbb{E}}_{x \sim \mathbb{F}_2^n} \left[\sum_{a \in \mathbb{F}_2} p(a, a, a | x, x, x) \right]$$

Strategies (cont.)

The **total variational distance** between two strategies p and q is

$$\|p - q\|_{\mathrm{TV}} = \mathop{\mathbb{E}}_{x,y,z \sim \mathbb{F}_2^n} \left[\sum_{a,b,c} |p(a,b,c|x,y,z) - q(a,b,c|x,y,z)| \right]$$

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- A small variational distance allows us to essentially "replace" one strategy with another

Quantum-soundness of the BLR

Now recall the 3 prover BLR test:

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- What if the provers have entangled strategies?

Quantum-soundness of the BLR (cont.)

- Instead of having functions f, g, h, Alice, Bob and Charlie now have measurements $\{A_x^a\}$, $\{B_y^b\}$ and $\{C_z^c\}$ with some shared entangled state $|\psi\rangle$.

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- A crucial reduction: given a strategy A, B, C, that wins with probability p, we can reduce down to a case where $A = B = C$, and have a winning probability of at least p.

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- A crucial reduction: given a strategy A, B, C, that wins with probability p, we can reduce down to a case where $A = B = C$, and have a winning probability of at least p.

  We call such a strategy **symmetric**.

Quantum-soundness of the BLR (cont.)

We still can!

Theorem (Ito-Vidick, 2012 [3])

Suppose three entangled provers succeed in the linearity test with probability 1-ϵ using a symmetric strategy $(|\psi\rangle, \{A_x^a\})$, and let its corresponding probability distributions be $\{p(a, b, c|x, y, z)\}$. Then there exists a classical linear strategy with shared randomness ℓ such that

$$\|p - \ell\|_{\mathrm{TV}} \leq 6\sqrt{3\epsilon^{1/2}}$$

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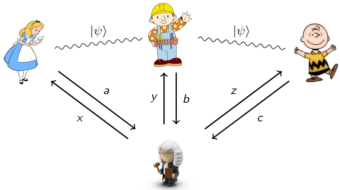
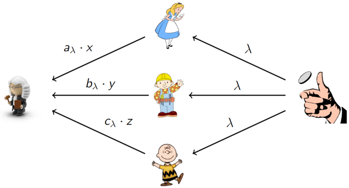


$\longleftrightarrow$



Our Results

We generalize the results of Ito-Vidick to $\mathbb{F}_p$:

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$$\|p - \ell\|_{\mathrm{TV}} \leq 6\epsilon^{1/4}\sqrt{1 + 2\left(1 - \frac{1}{p}\right)}^{1/2}$$

The Affine Linearity Test

Let's recall the BLR once more:

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Let provers A, B, C have some strategy. The verifier performs the following tests each with probability $1/2$:

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- If we drop this portion, can we still say anything about the strategies the players use?

The Affine BLR

Classically, we can! We can still show the strategies are close to some deterministic *affine* linear strategy:

Lemma

Given a classical probabilistic strategy p which succeeds in the linearity part of the BLR test with probability $1-\epsilon$, there exists a deterministic affine linear strategy ℓ such that

$$\|p-\ell\|_{\mathrm{TV}} \leq 2\left(1-\left(1-\sqrt{\epsilon}\right)^{3}\right)$$

- Can we draw the same conclusion with quantum strategies?

Our Results (cont.)

We generalize this result to the affine linear test in $\mathbb{F}_p$:

Theorem

Suppose three entangled provers succeed in the affine linearity test (i.e. linearity test without consistency) with probability 1-ϵ using a symmetric strategy $(\sigma, \{A_x^a\})$. Then there exists a classical affine linear strategy with shared randomness ℓ such that

$$\|p - \ell\|_{\mathrm{TV}} \leq 3\sqrt{2}\epsilon^{1/4}\sqrt{1 + 2\sqrt{2}\left(1 - \frac{1}{p}\right)^{1/2}}$$

Acknowledgements

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- and the MIT PRIMES organizers for their support

References

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